Suppose that $V$, $W$, $Z$ etc. are subspaces of $F^m$, $F^n$, $F^k$ etc. respectively, where $F$ is $\mathbb{R}$ or $\mathbb{C}$. As usual $\| \cdot \|_V$, $\| \cdot \|_W$, $\| \cdot \|_Z$, etc. denote the Euclidean norms on $V$, $W$, $Z$ etc. respectively.

1 **Notation.** We write $B_1$, $\overline{B}_1$ and $\partial B_1$ for the sets \( \{ x \in V \mid \|x\| < 1 \} \), \( \{ x \in V \mid \|x\| \leq 1 \} \) and \( \{ x \in V \mid \|x\| = 1 \} \) respectively, and refer to these as “the (Euclidean) unit ball”, “the closed unit ball” and “the unit sphere (of $V$)” respectively.

2 **Definition.** A function $F : V \to W$ is said to be “a Lipschitz function” if there exists a non-negative constant $\alpha$ such that
\[
\forall u, v \in V : \| F(u) - F(v) \|_W \leq \alpha \| u - v \|_V .
\]

3 **Fact.** It turns out that when $F$ is a Lipschitz function, there exists the smallest non-negative $\alpha$ for which (1) holds. This $\alpha$ is said to be “the Lipschitz constant” for $F$, and we denote it by $\mathcal{L}_F$.

It is obvious that $\mathcal{L}_F \geq 0$, and the equality holds if and only if $F$ is a constant function.

4 **Problem.**

1. Show that every linear function $A : V \to W$ is a Lipschitz function.

2. Show that if $A \in \mathcal{M}_n$ is a diagonal matrix, then $\mathcal{L}_A$ is the absolute value of the largest diagonal entry of $A$. 

We will discuss the proof of this fact in class. You can take for granted in the meantime.
3. Give a concrete example (with a full justification) to show that the result of part two becomes false if the words “diagonal matrix” are replaced with the words “triangular matrix”.

5 Problem. Suppose that $A, B : \mathcal{V} \to \mathcal{W}$ and $C : \mathcal{W} \to \mathcal{Z}$ are linear functions. Prove:

1. $\|\mathcal{L}_A\| \geq 0$, and the equality holds if and only if $A = 0$.

2. $\|\mathcal{L}_{aA}\| = |a| \|\mathcal{L}_A\|$.

3. $\|\mathcal{L}_{A+B}\| \leq \|\mathcal{L}_A\| + \|\mathcal{L}_B\|$, and give a concrete example (with a full justification) to show that the inequality may be strict.

4. $\|\mathcal{L}_{CA}\| \leq \|\mathcal{L}_C\| \|\mathcal{L}_A\|$, and give a concrete example (with a full justification) to show that the inequality may be strict.

6 Fact. Suppose that $A : \mathcal{V} \to \mathcal{W}$ is a linear function. Then there exists a unit vector $v \in \mathcal{V}$ such that

$$\|A(v)\| = \mathcal{L}_A.$$
2. Give a concrete example (with a full justification) to show that the result of part one becomes false if \( V = W_1 \oplus W_2 \oplus W_3 \oplus \ldots \oplus W_r \) is replaced with \( V = W_1 \oplus W_2 \oplus W_3 \oplus \ldots \oplus W_r \).

9 Problem. Suppose that \( A : V \longrightarrow W \) is a linear function.

1. Prove: \( \mathcal{L}_A \) is the maximal member of the set \( \{ |\langle A(x), y \rangle| \mid x, y \in \partial B_1 \} \).

2. Prove: \( \mathcal{L}_{(A^*)} = \mathcal{L}_A \).

3. Prove: If \( A \) is a non-zero partial isometry then \( \mathcal{L}_A = 1 \).

4. Prove: If \( B : V \longrightarrow V \) and \( C : W \longrightarrow Z \) are unitary linear functions, then \( \mathcal{L}_{(CAB)} = \mathcal{L}_A \).

5. Give a concrete example (with a full justification) to show that in general one does NOT expect similar (in a formal sense) linear transformations to have the same Lipschitz constants.

10 Problem.

1. Prove: If \( A : V \longrightarrow V \) is an orthodiagonalizable linear function, then

   (a) \( \mathcal{L}_A \) is the maximal member of the set \( \left\{ \frac{|\langle A(x), x \rangle|}{\langle x, x \rangle} \mid x \neq 0 \right\} \).

   (b) \( \mathcal{L}_A \) is the maximal member of the set \( \{ |\langle A(x), x \rangle| \mid x \in \partial B_1 \} \).

   (c) \( \mathcal{L}_A \) is the maximal element of the set \( \{ |\lambda| \mid \lambda \in \sigma_C(A) \} \).

2. Give a concrete example (with a full justification) to show that in a general (i.e. not necessarily orthodiagonalizable) case it may happen that \( \mathcal{L}_A \) is NOT the smallest upper bound for the sets \( \left\{ \frac{|\langle A(x), x \rangle|}{\langle x, x \rangle} \mid x \neq 0 \right\} \), \( \{ |\langle A(x), x \rangle| \mid x \in \partial B_1 \} \), and \( \{ |\lambda| \mid \lambda \in \sigma_C(A) \} \).

3. Prove: If \( B : V \longrightarrow W \) is a linear function then

\[
\mathcal{L}_{(B^*B)} = (\mathcal{L}_B)^2 = \mathcal{L}_{(BB^*)}.
\]