Mid-Term Take-Home Test, MA 353, S15

Out: Friday, March 13
Due: at 2 P.M. sharp on Friday, March 20, in class.

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1 Rules

This is to be a strictly individual effort.

You are not allowed to consult with any human being regarding this paper. Neither are you to consult any external sources of information (unless instructed to do so), with the exception of your notes, your textbooks, your assignments and the handouts. (Failure to adhere to these rules shall be considered academic dishonesty and will be reported to the administration of the College.)

You can use all of the information covered in class, including the results from the handouts, the assigned portions of the texts, and assignments 1-5, even if you did not get full marks for the solutions. Please indicate carefully what results you are quoting. If you appeal to a theorem, please don’t forget to check that all of the hypothesis of the theorem are satisfied. Do not use results which we have not covered in the course, unless you provide complete proofs of the results you are quoting, using only the concepts covered in our course so far.

Your take-home write-up must follow the same presentation and format (including stapling, etc.) rules as the assignments, and the grading scheme here is the same as that for the assignment problems. I will be particularly strict about sloppy write-ups and careless solutions.
Make careful choices of the problems to tackle first. All problems have the same weight, independent on how many parts they have. Go with quality over quantity, and give it your best shot. If you believe that a problem is incorrect as stated, make the necessary corrections and solve the corrected problem. (Of course, if the problem was correct and you were not, this would not be a good move...)

1 Fact. If one writes $r e^{i\theta}$ for $r \cos \theta + i r \sin \theta$, where $r \geq 0$ and $\theta \in \mathbb{R}$, then

$$re^{i\theta} pe^{i\beta} = r pe^{i(\theta+\beta)} \quad \text{and} \quad (re^{i\theta})^m = r^m e^{im\theta}.$$ 

2 Fact. Suppose that $\mathcal{V}$ is a subspace of $\mathbb{F}^n$, and $\mathcal{W}$ is a subspace of $\mathbb{F}^m$. Suppose that

$$\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2 \quad \text{and} \quad \mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2. \quad (1)$$

Then, for linear functions between $\mathcal{V}$ and $\mathcal{W}$, expressed as block-matrices with respect to the decompositions in (1), we have

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^* = \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix},$$

and the result may fail if the sums in (1) are not orthogonal.

3 Definition. Suppose that $\mathcal{V}$ is a subspace of $\mathbb{F}^n$, and $\mathcal{W}$ is a subspace of $\mathbb{F}^m$. Suppose that

$$\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2 \quad \text{and} \quad \mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2. \quad (2)$$

If $A_1 \in \mathcal{L}(\mathcal{V}_1, \mathcal{W}_1)$ and $A_2 \in \mathcal{L}(\mathcal{V}_2, \mathcal{W}_2)$, the direct sum $A_1 \oplus A_2$ of $A_1$ and $A_2$ is the element of $\mathcal{L}(\mathcal{V}, \mathcal{W})$ defined by:

$$(A_1 \oplus A_2)(v_1 + v_2) \overset{\text{def}}{=} A_1(v_1) + A_2(v_2), \quad \text{for} \quad v_i \in \mathcal{V}_i, \quad w_i \in \mathcal{W}_i;$$

in other words, $A_1 \oplus A_2$ is the function

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix},$$

expressed as a block matrix with respect to the decompositions in (2).

If BOTH of the direct sums in (2) are orthogonal, we write $A_1 \oplus A_2$ (instead of $A_1 \oplus A_2$), and refer to it as an ortho-direct sum of $A_1$ and $A_2$. 

2
2 Problems

4 Problem. Prove or disprove each of the following claims.

1. A composition of two isometries is an isometry.

2. A composition of two co-isometries is a co-isometry.

3. A composition of two partial isometries is a partial isometry.

4. An ortho-direct sum of two isometries is an isometry.

5. An ortho-direct sum of two co-isometries is a co-isometry.

6. An ortho-direct sum of two partial isometries is a partial isometry.

5 Problem. Suppose that \( V \) is a subspace of \( \mathbb{F}^n \) and \( W \) is a subspace of \( \mathbb{F}^m \). Prove that every partial isometry \( A \in \mathcal{L}(V, W) \) can be extended to a \( B \in \mathcal{L}(V, W) \) which is either an isometry or a co-isometry.

6 Problem. Prove or disprove each of the following claims.

1. For a subspace \( V \) of \( \mathbb{R}^n \), and \( A \in \mathcal{L}(V) \) the following are equivalent:

   (a) \( A \) is ortho-diagonalizable;

   (b) There is a polynomial \( p \) with Real coefficients such that \( A^* = p(A) \).

2. For a subspace \( V \) of \( \mathbb{C}^n \), and \( A \in \mathcal{L}(V) \) the following are equivalent:

   (a) \( A \) is ortho-diagonalizable;

   (b) There is a polynomial \( p \) with Complex coefficients such that \( A^* = p(A) \).

7 Problem. Suppose that \( V \) is a subspace of \( \mathbb{F}^n \), \( A \in \mathcal{L}(V) \) is positive semi-definite. Show that for each \( k \in \mathbb{N} \) there is a UNIQUE positive semidefinite \( P \in \mathcal{L}(V) \) such that

\[
p^k = A.
\]
8 Problem. Suppose that \( V \) is a subspace of \( F^n \), \( A \in \mathcal{L}(V) \) is positive semi-definite.

1. Prove that for every orthonormal basis \( \beta \) of \( V \):
\[
\left[ \sqrt{A} \right]_{\beta \leftarrow \beta} = \sqrt{[A]_{\beta \leftarrow \beta}}.
\]

2. Give a concrete example to show that the above formula can fail for a non-orthonormal basis \( \beta \).

9 Problem. Give an example, if such exists, of a subspace \( V \) of the vector space \( Q^n \) (over the field \( Q \) of the rational numbers) with \( \dim V \geq 3 \), and an operator \( A \in \mathcal{L}(V) \) such that
\[
\text{Lat}(A) = \{V, \{0_n\}\}.
\]

Don’t forget to justify your claims. If no such example exists, prove that this is so.

10 Problem. Suppose that \( V \) is a subspace of \( R^n \), and that \( A \in \mathcal{L}(V) \) is normal. Prove or disprove the following claim:

The following are equivalent for \( B \in \mathcal{L}(V) \):

1. \( B \) commutes with \( A \);

2. \( B \) commutes with \( A^* \).

11 Problem (Simultaneous Schur’s Theorem over \( C \)). Suppose that \( V \) is a subspace of \( C^n \), and that \( C \subset \mathcal{L}(V) \) is an abelian* collection. Prove that there is an orthonormal basis \( \beta \) of \( V \) such that
\[
\forall A \in C : \ [A]_{\beta \leftarrow \beta} \text{ is upper triangular}.
\]

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*i.e. all elements of the collection commute with each other.*