1 “Under Competent Interrogation Each Basis Will Sing Like a Canary”.

**1 Review.** Let us recall that for each coordinate system $\Omega = \{U_1, U_2, U_3, \ldots, U_n\}$ of a vector space $V$ we have constructed a corresponding isomorphism $\Gamma_\Omega : \mathbb{R}^n \rightarrow V$ defined by

$$\Gamma_\Omega \left( \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{array} \right) \overset{\text{def}}{=} x_1 U_1 + x_2 U_2 + \ldots + x_n U_n.$$ 

The inverse of $\Gamma_\Omega$ is the isomorphism $\left[ \cdot \right]_\Omega : V \rightarrow \mathbb{R}^n$. Clearly, for every $V \in V$:

$$\left[ V \right]_\Omega = \left( \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{array} \right), \text{ such that } V = a_1 U_1 + a_2 U_2 + \ldots + a_n U_n.$$ 

Since obviously $\Gamma_\Omega (E_i) = U_i$, we also have

$$\left[ U_i \right]_\Omega = E_i.$$
In other words,

The isomorphism \([ \cdot ]_\Omega\) maps the coordinate system \(\Omega\) (that had generated it) to the standard basis of \(\mathbb{R}^n\), and, of course, \(\Gamma_\Omega\) maps the standard basis of \(\mathbb{R}^n\) to \(\Omega\).

2 Test Your Comprehension. Since \([ \cdot ]_\Omega\) and \(\Gamma_\Omega\) are isomorphisms, they map coordinate systems to coordinate systems. Give a precise argument to show that every coordinate system of \(V\) is an image under \(\Gamma_\Omega\) of a coordinate system of \(\mathbb{R}^n\), and vice versa.

Since we have a good handle on the coordinate systems of \(\mathbb{R}^n\) (these being the columns of invertible matrices), once we have just one coordinate system of \(V\), we can generate all coordinate systems of \(V\):

3 Theorem. If \(\Omega = \{U_1, U_2, U_3, \ldots, U_n\}\) is a coordinate system of a linear space \(V\), then the following are equivalent

1. \(\Gamma = \{V_1, V_2, V_3, \ldots, V_n\}\) is (another) coordinate system of \(V\);
2. \([V_1]_\Omega, [V_2]_\Omega, [V_3]_\Omega, \ldots, [V_n]_\Omega\) is a coordinate system of \(\mathbb{R}^n\);
3. The matrix \([V_1]_\Omega [V_2]_\Omega [V_3]_\Omega \ldots [V_n]_\Omega\) is invertible;
4. \(\text{Det}_n \begin{bmatrix} [V_1]_\Omega & [V_2]_\Omega & [V_3]_\Omega & \ldots & [V_n]_\Omega \end{bmatrix} \neq 0\).

4 Example. We have seen that \(\Omega = \{1, x, x^2, x^3\}\) is a basis of the linear space \(P_3[x]\) of all polynomial functions of degree at most 3. Consider the set

\[\Delta = \{1 - 3x + 4x^2, \ 3 - 2x + 7x^2 - 5x^3, \ 7 + x^2 - 3x^3, \ 6 + 3x - 8x^2 - 9x^3\}\]

Is \(\Delta\) a coordinate system of \(P_3[x]\)?
To answer this question we use Theorem 3 and consider
\[ \{ [1 - 3x + 4x^2]_\Omega, [3 - 2x + 7x^2 - 5x^3]_\Omega, [7 + x^2 - 3x^3]_\Omega, [6 + 3x - 8x^2 - 9x^3]_\Omega \}, \]
which of course equals
\[ \begin{cases} \begin{pmatrix} 1 \\ -3 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ 7 \\ -5 \end{pmatrix}, \begin{pmatrix} 7 \\ 0 \\ 1 \\ -3 \end{pmatrix}, \begin{pmatrix} 6 \\ 3 \\ -8 \\ -9 \end{pmatrix} \end{cases}. \]

Using Mathematica we can see that
\[ \text{Det}_4 \begin{pmatrix} 1 & 3 & 7 & 6 \\ -3 & -2 & 0 & 3 \\ 4 & 7 & 1 & -8 \\ 0 & -5 & -3 & -9 \end{pmatrix} = 924 \neq 0. \]
Thus indeed \( \Delta \) is a coordinate system of \( P_3[x] \).

2 Matrices Corresponding to Linear Transformations

5 Notation. If \( \Omega = \{ U_1, U_2, U_3, \ldots, U_n \} \) is a coordinate system of a linear space \( \mathcal{V} \), and \( T : \mathcal{V} \xrightarrow{\text{linear}} \mathcal{V} \), then
\[ [\ ]_\Omega \circ T \circ \Gamma_\Omega : \mathbb{R}^n \rightarrow \mathbb{R}^n \]
is a composition of three linear functions, and thus is itself linear, and hence a matrix!!!
We shall denote this matrix by \([T]_{\Omega\rightarrow\Omega}\). So:
\[ [T]_{\Omega\rightarrow\Omega} \stackrel{\text{def}}{=} [\ ]_\Omega \circ T \circ \Gamma_\Omega = [\ ]_\Omega \circ T \circ [\ ]_\Omega^{-1} = \Gamma_\Omega^{-1} \circ T \circ \Gamma_\Omega. \]

6 Notation. Given a linear space \( \mathcal{W} \), we shall write \( \mathcal{L}(\mathcal{W}) \) for the set of all linear functions \( L : \mathcal{W} \rightarrow \mathcal{W} \). It is easy to check that \( \mathcal{L}(\mathcal{W}) \) is itself a linear space when it is equipped with the usual addition and scaling of functions.
7 Exercise. Suppose that \( \Omega = \{ U_1, U_2, U_3, \ldots, U_n \} \) is a coordinate system of a linear space \( V \), and \( T, S : V \xrightarrow{\text{linear}} V \). Prove each of the following:

1. \([ T \circ S ]_{\Omega \rightarrow \Omega} = [ T ]_{\Omega \rightarrow \Omega} \circ [ S ]_{\Omega \rightarrow \Omega} ;\)
2. \([ \alpha T + \beta S ]_{\Omega \rightarrow \Omega} = \alpha [ T ]_{\Omega \rightarrow \Omega} + \beta [ S ]_{\Omega \rightarrow \Omega} ;\)
3. The function \([ \ ]_{\Omega \rightarrow \Omega} : \mathcal{L}(V) \rightarrow \mathbb{M}_n \), that maps each linear function \( T \) to the matrix \([ T ]_{\Omega \rightarrow \Omega} \), is an isomorphism;
4. \([ T(V) ]_{\Omega} = [ T ]_{\Omega \rightarrow \Omega} ([ V ]_{\Omega}) \), for all \( V \in V \).

As a consequence of part 4 of Exercise 7, and Review 1, we see that

\[ [ T(U_i) ]_{\Omega} = [ T ]_{\Omega \rightarrow \Omega} ([ U_i ]_{\Omega}) \]
\[ = [ T ]_{\Omega \rightarrow \Omega} (E_i) \]
\[ = \text{ the } i\text{-th column of } [ T ]_{\Omega \rightarrow \Omega} \]

Another way to reach the same conclusion is as follows:

the \( i \)-th column of \([ T ]_{\Omega \rightarrow \Omega} = [ T ]_{\Omega \rightarrow \Omega} (E_i) \)
\[ = [ \ ]_{\Omega} \circ T \circ \Gamma (E_i) \]
\[ = [ \ ]_{\Omega} \circ T (U_i) \]
\[ = [ T(U_i) ]_{\Omega} \]

We now have the following “recipe” for constructing the matrix \([ T ]_{\Omega \rightarrow \Omega} \) associated with \( T \):

the \( i \)-th column of \([ T ]_{\Omega \rightarrow \Omega} \) is \([ T(U_i) ]_{\Omega} \)

8 Example. Suppose we consider the basis \( \Delta \) of \( P_3([x]) \) presented in Example 4, and let us take \( T \) to be the differentiation function. We already know that
$T \in \mathcal{L}(P_3[x])$. Our task here is to find $[T]_{\Delta \rightarrow \Delta}$.

\[
(\text{the 1-st column of } [T]_{\Delta \rightarrow \Delta}) = [T(1 - 3x + 4x^2)]_{\Delta} = [-3 + 8x]_{\Delta},
\]

and here we run into a challenge: how do we evaluate $[-3 + 8x]_{\Delta}$? In other words how do we find the (unique) coefficients $a_1, a_2, a_3, a_4$ such that

\[
-3 + 8x = a_1(1 - 3x + 4x^2) + a_2(3 - 2x + 7x^2 - 5x^3) + a_3(7 + x^2 - 3x^3) + a_4(6 + 3x - 8x^2 - 9x^3)? \tag{1}
\]

One method is to use “brute force”. Two polynomials are equal exactly when they have equal coefficients in front of equal powers of the variable. Thus, for (1) to be true we should have

\[
\begin{align*}
-3 &= 1a_1 + 3a_2 + 7a_3 + 6a_4 \\
8 &= 3a_1 - 2a_2 + 3a_3 + 3a_4 \\
0 &= 4a_1 + 7a_2 + a_3 - 8a_4 \\
0 &= 0a_1 - 5a_2 - 3a_3 - 9a_4
\end{align*}
\]

which we can rewrite in matrix form as:

\[
\begin{pmatrix}
-3 \\
8 \\
0 \\
0
\end{pmatrix} = \begin{pmatrix}
1 & 3 & 7 & 6 \\
-3 & -2 & 0 & 3 \\
4 & 7 & 1 & -8 \\
0 & -5 & -3 & -9
\end{pmatrix} \begin{pmatrix}
a_1 \\
a_2 \\
a_3 \\
a_4
\end{pmatrix}. \tag{2}
\]

This is remarkable! The matrix involved is exactly the matrix we have looked at in Example 4! Coincidence? I think NOT! But there is a bit of a story to tell here.

(In the meantime, I hope you already know how to use Mathematica to solve (2) for the $a$’s. This can be done because as we have seen in Example 4, the matrix involved in (2) is invertible.)

Let us write $p_1, p_2, p_3, p_4$ for $1 - 3x + 4x^2$, $3 - 2x + 7x^2 - 5x^3$, $7 + x^2 - 3x^3$, and $6 + 3x - 8x^2 - 9x^3$ respectively, so that $\Delta = \{p_1, p_2, p_3, p_4\}$. Then
equation (2) can be expressed this way:

\[ [T(p_1)]_\Omega = \begin{bmatrix} [p_1]_\Omega & [p_2]_\Omega & [p_3]_\Omega & [p_4]_\Omega \end{bmatrix} \begin{bmatrix} (T(p_1))_\lambda \end{bmatrix} \]

In other words, somehow the matrix

\[ \begin{bmatrix} [p_1]_\Omega & [p_2]_\Omega & [p_3]_\Omega & [p_4]_\Omega \end{bmatrix} \]

“converts” \([T(p_1)]_\lambda\) into \([T(p_1)]_\Omega\).

It turns out that this is true more generally, in the following sense,

\[ [w]_\Omega = \begin{bmatrix} [p_1]_\Omega & [p_2]_\Omega & [p_3]_\Omega & [p_4]_\Omega \end{bmatrix} \begin{bmatrix} (w) \end{bmatrix} \]

for every \(w \in P_3[x]\).

In fact, this sort of thing is a general theorem for linearspaces other than \(P_3[x]\). Here is what it states:

\section*{9 Theorem.} If \(\beta = \{V_1, V_2, V_3, \ldots, V_n\}\) and \(\gamma = \{U_1, U_2, U_3, \ldots, U_n\}\) are coordinate systems of a linear space \(V\) then

\[ [W]_\gamma = \begin{bmatrix} [V_1]_\gamma & [V_2]_\gamma & [V_3]_\gamma & \ldots & [V_n]_\gamma \end{bmatrix} \begin{bmatrix} (W)_{\beta} \end{bmatrix}, \]

for every \(W \in V\).

For this reason, the matrix

\[ \begin{bmatrix} [V_1]_\gamma & [V_2]_\gamma & [V_3]_\gamma & \ldots & [V_n]_\gamma \end{bmatrix} \]

is said to be a change of basis matrix for the switch “\(\gamma \leftarrow \beta\)” of coordinate systems, and

it is denoted by \([id]_{\gamma \leftarrow \beta}\).

(The rational for this notation shall become obvious shortly.)

\section*{Proof.} Under our usual set-up and notation, consider the function

\[ \varphi \overset{\text{def}}{=} [.]_\gamma \circ \Gamma_\beta. \]

Note that \(\varphi : \mathbb{R}^n \xrightarrow{\text{isomorphism}} \mathbb{R}^n\), since it is a composition of two isomorphisms; i.e. \(\varphi\) is an invertible \(n \times n\) matrix. The rest of the proof is left as a (following) exercise.
10 Exercise.

1. Show that \([W]_\gamma = \varphi \left( [W]_\beta \right)\), for any \(W \in \mathcal{V}\).

2. Show that \(\varphi = \left[ [V_1]_\gamma \ [V_2]_\gamma \ [V_3]_\gamma \ \ldots \ [V_n]_\gamma \right]\)
   
   \text{Hint: Think columns. The } i\text{-th column of } \varphi \text{ equals } \varphi(\ldots)

3. Explain why the observation
   \[
   \varphi \overset{\text{def}}{=} \left[ \right]_\gamma \circ \Gamma_\beta = \left[ \right]_\gamma \circ id_V \circ \Gamma_\beta
   \]
   naturally leads to the notation \([id_V]_{\gamma \leftarrow \beta}\) for \(\varphi\).

\[\square\]

11 Exercise. Suppose that \(\beta = \{V_1, V_2, V_3, \ldots, V_n\}\) and \(\gamma = \{U_1, U_2, U_3, \ldots, U_n\}\)
are bases of a linear space \(\mathcal{V}\), and \(T : \mathcal{V} \to \mathcal{V}\). Prove:

1. \(\left( [id_V]_{\gamma \leftarrow \beta} \right)^{-1} = [id_V]_{\beta \leftarrow \gamma} \)

2. \([T]_{\gamma \leftarrow \gamma} = [id_V]_{\gamma \leftarrow \beta} [T]_{\beta \leftarrow \beta} [id_V]_{\beta \leftarrow \gamma} \)

12 Exercise. Find \([T]_{\Delta \leftarrow \Delta}\) in Example 8, by finding \([T]_{\Omega \leftarrow \Omega}\) first, and then using
the result of the Exercise 11.

13 Exercise. Suppose that \(\mathcal{V}\) is an \(n\)-dimensional linear space. Show that

\[
\left\{ [id]_{\beta \leftarrow \gamma} \mid \beta \text{ and } \gamma \text{ are basis of } \mathcal{V} \right\} = \left\{ A \in \mathbb{M}_n \mid A \text{ is invertible in } \mathbb{M}_n \right\}.
\]

\text{Hint: We already know that the inclusion } \subseteq \text{ holds (why?). Thus we only need to show that the reverse inclusion } \supseteq \text{ holds. To this end, start with an arbitrary invertible } A \in \mathbb{M}_n. \text{ Pick and fix a coordinate system } \beta_0 \text{ of } \mathcal{V}. \text{ Now construct a coordinate system } \gamma \text{ such that } [id]_{\beta_0 \leftarrow \gamma} = A. \text{ (Do some “diagram chasing”.)} \]